# Forecasting US Treasury Yield Curve Using The Cox-Ingersoll-Ross (CIR) Model

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# 1 Introduction

Interest rates and yield curve play an important role in the financial markets. It affects the prices of different financial instruments such as derivatives. It also affects economic decisions such as monetary policies and debt policies. The commonly used benchmark interest rates in the market are derived from the yields of treasury bills and treasury bonds issued by the government. These are generally considered "risk-free" instruments since governments have a low probability of default. Furthermore, the yield curves derived from these instruments are typically upward sloping curves which reflects the premium for greater exposure to market risk for longer maturities. Nevertheless, it is also important to note that the shape of yield curves change throughout time. It moves up and down and changes shape which are typically monotonic, humped, or S shaped.

Suppose at time $t$, the value of a zero-coupon bond with maturity date $T$ is $P(t,T)$. Using this instrument, several types of interest rates can be derived. The forward rate is the defined as the riskless interest rate over the future interval $[S,T]$. This can be obtained by selling one unit of $S$-bond at time $t$ and using the proceeds to buy $\dfrac{P(t,S)}{P(t,T)}$ units of $T$-bonds. Hence, the continuously compounded forward rate for the interval $[S,T]$ contracted at time $t$ is given by

$$R(t;S,T) = -\frac{\ln P(t,T) - \ln P(t,S)}{T-S}.$$

Likewise, the instantaneous forward rate is defined as the riskless interest rate over the (future) infinitesimally small interval $[T, T+dT]$. Mathematically, this is the limit of the continuously compounded forward rate as $S \to T$. Hence, the instantaneous forward rate with maturity date $T$ contracted at time $t$ is given by

$$f(t,T) = \lim_{S \to T} R(t;S,T) = -\frac{\partial}{\partial T} \ln P(t,T).$$

Finally, the short rate is defined as the riskless interest rate over an infinitesimally small interval $[t, t+dt]$. Hence, the short rate at time $t$ is given by

$$r_t = f(t,t).$$

The prices of several financial instruments such as bonds, options, and other derivatives depends on the process $r_t$. Moreover, as opposed to stock prices, short rates tend to stay within a certain range. Hence, they are mean-reverting process. It is also worth noting that in practice, the process $r_t$ is not directly observable in the market. The shortest maturity available is the overnight rate, which is conceptually different from the short rate. Nevertheless, in practice, the overnight, one-night, and three-month rates are used as proxies for $r_t$.

Due to the importance of the short rate in the pricing of different financial instruments, several models in the literature attempt to capture the short rate (stochastic) process. One-factor models are a commonly used models to capture the dynamics of the short rate process in a parsimonious manner. In these models, the process for $r_t$ involves only one source of uncertainty. The process is also assumed to be stationary. This paper explores a one-factor model for short rate – the Cox-Ingersoll-Ross (CIR) model.

## 2 Conceptual Discussion

The Cox-Ingersoll-Ross (CIR) model (1985) is a one-factor Markov model, which proposes an improvement to the Vasicek model. The Vasicek model (1977) was the first short rate model to capture the mean-reversion property of interest rates (Hull, n.d.). The risk-neutral process for $r_t$ under the Vasicek model is given by

$$dr_t = a\left(b - r_t\right)dt + \sigma dW_t,$$

where $a$, $b$, and $\sigma$ are nonnegative constants and $W_t$ is a standard Wiener process. This model incorporates the mean reversion property, wherein the short rate is pulled to a reversion level $b$ at a reversion rate $a$. To illustrate, the drift term is positive when $r_t < b$ to push the short rate up. On the other hand, the drift term is negative when $r_t > b$ to push the short rate down. The normally distributed uncertain component of the SDE above, given by $\sigma dW_t$, determines the variation of the forecasts of the short rate process.

Furthermore, the price of the zero-coupon bond under the Vasicek model is

$$P\left(t,T\right) = A(t,T)e^{-B(t,T)r_t}$$

where

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a},$$

$$A(t,T) = \exp\left[\frac{\left(B(t,T) - T + t\right)\left(a^2 b - \sigma^2/2\right)}{a^2} - \frac{\sigma^2 B(t,T)^2}{4a}\right].$$

However, the Vasicek model allows the short rate $r_t$ to be negative, which is undesirable in an economy.

The Cox-Ingersoll-Ross (CIR) model (1985) introduced a "square root" term in the diffusion coefficient of the short rate dynamics so that $r_t$ can never be negative. Hence, the CIR model is able to capture the mean-reverting property of interest rates as well as the nonnegativity of interest rates, provided that $2ab \geq \sigma^2$ (Hull, n.d.). The CIR model is given by

$$dr_t = a\left(b - r_t\right)dt + \sigma\sqrt{r_t}dW_t,$$

where $a$, $b$, and $\sigma$ are nonnegative constants. This model captures the same mean-reverting drift as the Vasicek model, but the standard deviation of the change in the short rate in a short period of time is proportional to $\sqrt{r_t}$. This implies that as the short-term interest rate increases, the volatility (standard deviation) increases as well.

Likewise, the price of a zero-coupon bond under the CIR model in the risk-neutral world follows the same general form as the Vasicek model,

$$P\left(t,T\right) = A(t,T)e^{-B(t,T)r_t}.$$

However, the functions $B(t,T)$ and $A(t,T)$ are given as

$$B(t,T) = \frac{2\left(e^{\gamma(T-t)} - 1\right)}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma},$$

$$A(t,T) = \left[\frac{2\gamma e^{(a+\gamma)(T-t)/2}}{(\gamma+a)\left(e^{\gamma(T-t)}-1\right)+2\gamma}\right]^{2ab/\sigma^2},$$

where $\gamma = \sqrt{a^2 + 2\sigma^2}$.

For both the Vasicek and CIR model, the price of a zero-coupon bond follows the form

$$P(t,T) = A(t,T)e^{-B(t,T)r_t}.$$

Hence,

$$\frac{\partial P(t,T)}{\partial r_t} = -B(t,T)P(t,T).$$

Moreover, the continuously compounded zero rate at time $t$ for a period of $(T-t)$ is linearly dependent on $r_t$ and is given by

$$\begin{aligned} R(t,T) &= -\frac{1}{T-t}\ln P(t,T) \\ &= -\frac{1}{T-t}\ln A(t,T) + \frac{1}{T-t}B(t,T)r_t. \end{aligned}$$

Thus, the entire yield curve at time $t$ can be determined as a function of $r_t$ once $a$, $b$, and $\sigma$ have been fixed. Additionally, the shape of term structure at a particular time under this model can be upward sloping, downward sloping, or slightly "humped."

In the real-world setting, bonds have a positive market price of risk, so investors require an extra return over the risk-free rate to compensate for the risks in investing in bonds. Suppose that $\lambda$ is the negative market price of risk of the short rate. If the risk neutral process under the CIR model is given by

$$dr_t = a(b - r_t)\,dt + \sigma\sqrt{r_t}dW_t,$$

and assuming the market price of risk $\lambda = \kappa\sqrt{r_t}$, where $\kappa$ is a negative constant, then the real-world process under the CIR model becomes

$$\begin{aligned} dr_t &= [a(b-r_t) + \kappa\sigma r_t]\,dt + \sigma\sqrt{r_t}dW_t \\ &= a^*(b^* - r_t)\,dt + \sigma\sqrt{r_t}dW_t, \end{aligned}$$

where $a^* = a - \kappa\sigma$ and $b^* = \dfrac{ab}{a^*}$. To get the functions $A(t,T)$ and $B(t,T)$ in the real-world, $a$ and $b$ are replaced with $a^*$ and $b^*$, respectively. Hence, the real-world process is the same as the risk-neutral process except that the reversion rate is higher and the reversion level is lower.

Following Miao (2018), in estimating the parameters $a$, $b$, and $\sigma$, the discrete version of the CIR model is considered. Suppose $t_i$ is the $i$th trading day of the period $[0,T]$. Then, the short rate follows the process

$$\begin{aligned} \Delta r_t &= a^*(b^* - r_t)\,\Delta t + \sigma\sqrt{|r_t|}\Delta W_t \\ r_{t+1} - r_t &= a^*(b^* - r_t)\,\Delta t + \sigma\sqrt{|r_t|}\sqrt{\Delta t}\;\varepsilon, \end{aligned}$$

where $\varepsilon \sim N(0,1)$. Dividing both sides by $\sqrt{|r_t|}$,

$$\begin{aligned}\frac{r_{t+1}-r_t}{\sqrt{|r_t|}} &= \frac{a^*\left(b^*-r_t\right)\Delta t}{\sqrt{|r_t|}} + \sigma\sqrt{\Delta t}\ \varepsilon \\ &= a^*b^*\frac{\Delta t}{\sqrt{|r_t|}} - a^*\sqrt{|r_t|}\Delta t + \sigma\sqrt{\Delta t}\ \varepsilon \\ &= \beta_1 x_1 + \beta_2 x_2 + \epsilon,\end{aligned}$$

where $\beta_1 = a^*b^*$, $\beta_2 = a^*$, $x_1 = \frac{\Delta t}{\sqrt{|r_t|}}$, and $x_2 = -\sqrt{|r_t|}\Delta t$. In this form, it is apparent that the coefficients $a$, $b$, $\sigma$ can be obtained using linear regression, where $\sigma = \frac{\text{std error}}{\sqrt{\Delta t}}$.

After obtaining the parameter values, the short rate can then be forecasted using the discretized CIR model. Given $r_t$, the short rate tomorrow $r_{t+1}$ is obtained as

$$r_{t+1} = r_t + a^*\left(b^*-r_t\right)\Delta t + \sigma\sqrt{|r_t|}\sqrt{\Delta t}\ \varepsilon,$$

where $\varepsilon \sim N(0,1)$ is a randomly generated number from the standard normal distribution.

# 3 Results and Discussion

The interest rate data considered in this paper is the US Treasury Par Yield Curve Rates. The period considered is from January 2018 to December 2018, where today is assumed to be December 31, 2018. It is worth noting that the interest rates for maturities less than or equal to 1 year are considered as zero rates. However, interest rates for maturities greater than 1 year are not zero rates. Nevertheless, zero rates can be easily obtained using the bootstraping method.

Let $n_j$ be the number of years corresponding to tenor $j$. First, interest rates for increments of 0.5 years are obtained using linear interpolation. Afterwards, the discount factors $\mathrm{DF}_{n_j}$ are obtained. For $j \leq 1$,

$$\mathrm{DF}_{n_j} = \frac{1}{1+r_j n_j}.$$

For $j > 1$,

$$\mathrm{DF}_{n_j} = \frac{1 - r_j \sum_{i=1}^{2j-1}\left(n_{0.5i} - n_{0.5(i-1)}\right) \times \mathrm{DF}_{n_{0.5i}}}{1 + \left(n_j - n_{j-0.5}\right) \times r_j}.$$

Finally, the zero rates are obtained as

$$z_{n_j} = -\frac{\ln\left(\mathrm{DF}_{n_j}\right)}{n_j}.$$

For simplicity, a 30/360 day count convention is used.

After obtaining the yield curve data, the 3M interest rate is considered as a proxy for the short rate (Chapman et al., 1999). Since the quoted rates are simple interest rates, it is converted into continuously compounding interest rates using the formula

$$z_{n_j} = \frac{\ln(1 + r_j n_j)}{n_j},$$

where $r_j$ is the simple interest rate. Table 1 below shows the 3M interest rates used as proxy for the short rate.

| Date | 3M (Simple) | 3M (Continuous) |
|---|---|---|
| 2018-01-02 | 1.4400 | 1.4374 |
| 2018-01-03 | 1.4100 | 1.4075 |
| 2018-01-04 | 1.4100 | 1.4075 |
| 2018-01-05 | 1.3900 | 1.3876 |
| 2018-01-08 | 1.4500 | 1.4474 |
| 2018-01-09 | 1.4400 | 1.4374 |
| 2018-01-10 | 1.4200 | 1.4175 |

Table 1: 3M Interest Rates

Next, a linear regression is performed to obtain the parameters $a$, $b$, $\sigma$. Using the formula presented above, the results obtained are presented in Table 2. For simplicity $\Delta t$ is assumed to be $\frac{1}{260}$, assuming 260 trading days in a year.

| Coefficient | Value |
|---|---|
| $\beta_1$ | 0.028595 |
| $\beta_2$ | 0.919043 |
| std error | 0.001416 |
| $a^*$ | 0.919043 |
| $b^*$ | 0.031114 |
| $\sigma$ | 0.022839 |

Table 2: Linear Regression Results

Hence, the CIR model is given as

$$r_{t+1} = r_t + 0.919043\,(0.031114 - r_t)\,\frac{1}{260} + 0.022839\sqrt{|r_t|}\sqrt{\frac{1}{260}}\;\varepsilon.$$

Using this, the daily short rate for the period January 2019 to March 2019 is obtained. Furthermore, the yield curve for each day is also obtained using the formula presented above. The interest rates for maturities 1M, 2M, 3M, 6M, 1Y, 2Y, 3Y, 5Y, 7Y, 10Y, 20Y, and 30Y are obtained. 1,000 simulations are considered and the average is taken. Figure 1 presents the actual short rate (3M interest rate) and the forecasted short rate for the period January 2019 to March 2019 using the CIR model. It is worth noting that the CIR model performs poorly in predicting short rates as the predicted short rate differs widely from the actual short rate.

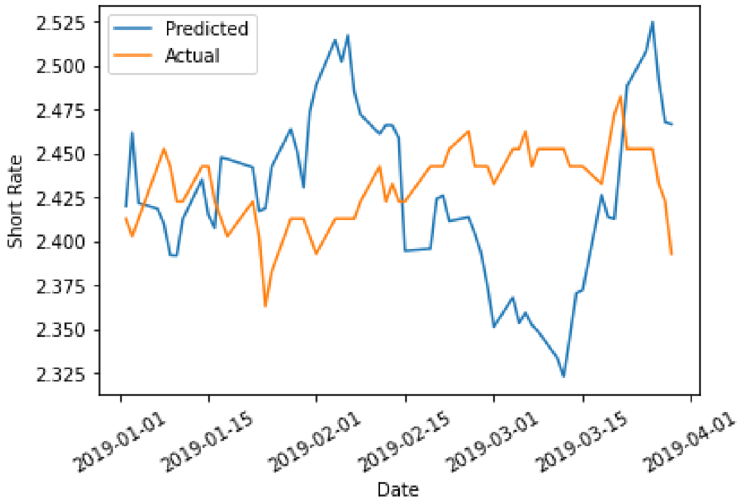


Figure 1: Actual vs Forecasted Short Rate

Likewise, Figures 2 to 3 shows the actual yield curve and forecasted yield curve for the period January 2019 to March 2019. It can be observed that the shape of the forecasted yield curve is very different from the actual yield curve for the period.

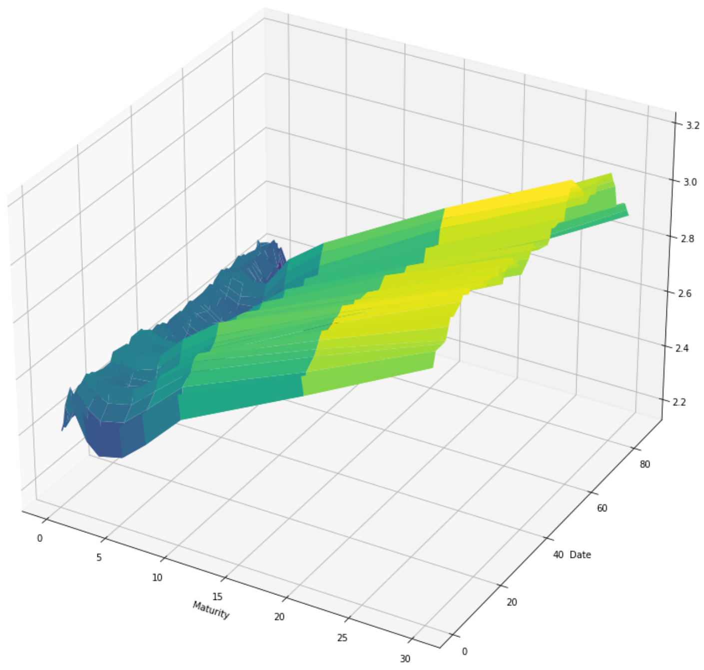


Figure 2: Actual Yield Curve

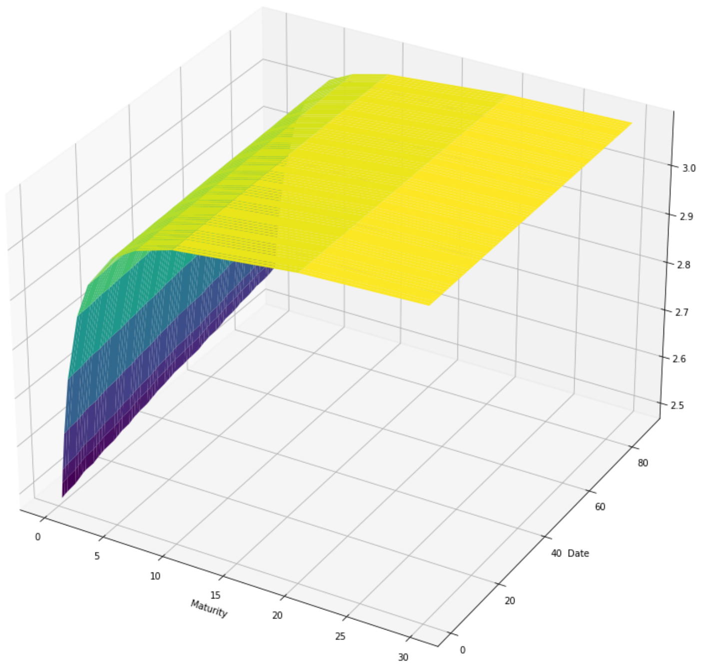


Figure 3: Forecasted Yield Curve

As a closer analysis, the actual and forecasted yield curves for March 29, 2019 are shown in Figure 4. It is worth noting that the shapes of the yield curves are different. The actual yield curve decreases in the short-term and increases in the long-term. However, the forecasted yield curve monotonically increases as the maturity increases.

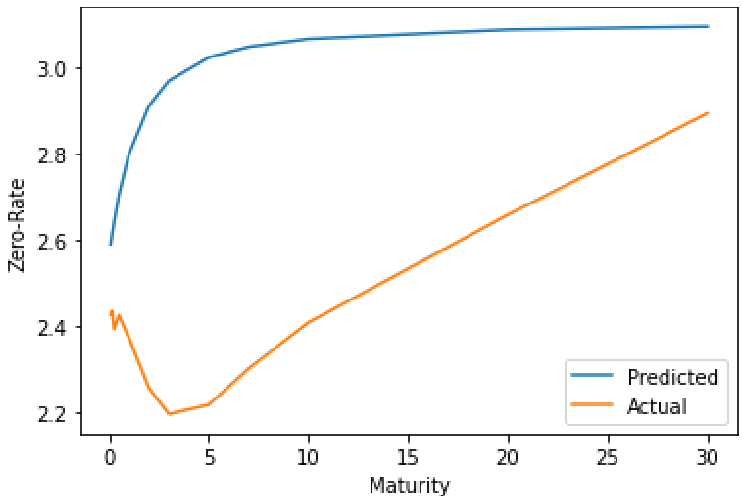


Figure 4: March 29, 2019 Yield Curves

Overall, the results show that the CIR model performs poorly in forecasting the US Treasury yield curve. The shape of the yield curve under the CIR model are upward sloping, downward sloping, or slightly "humped." However, the actual yield curve exhibits a shape outside the scope of the model, which can explain the poor performance.

# 4 Conclusion

The Cox-Ingersoll-Ross (CIR) model (1985) is a useful model in forecasting short rates. It is able to capture both the mean-reversion property and nonnegativity of interest rates which makes it an attractive model. Moreover, the model has an affine term structure, making it possible to derive a yield curve from the short rate. While the results obtained highlighted the weakness of the model in forecasting the short rates and yield curves, further studies can improve its performance by considering a longer in-sample period to accommodate a wide variety of yield curves. Moreover, a different period can also be considered. As shown in Figure 5, the yield curve for the in-sample period from January 2018 to December 2018 is mostly monotonically increasing similar to the forecasted yield curve. Hence, there could have been an economic event during the out-of-sample period from January 2019 to March 2019 that could explain the sudden change in the shape of the yield curve.

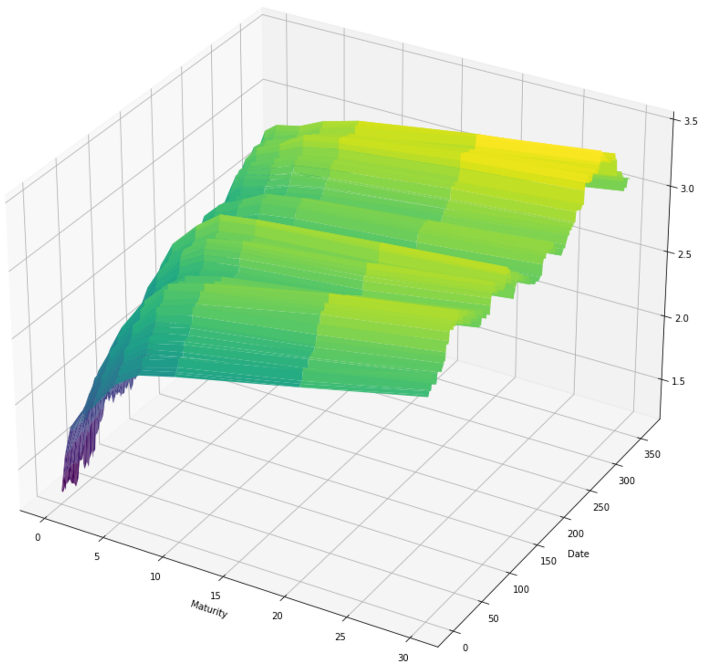


Figure 5: In-Sample Yield Curve

# References

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