Valuation of Reset Put Option

Chelsea Go, Brian Lim

Department of Mathematics
School of Science and Engineering
Ateneo de Manila University

Introduction

Introduction

Reset Put Option

A reset put option is similar to a standard put option except that the exercise price is reset equal to the stock price on the pre-specified reset date if this stock price exceeds the original exercise price.

Unlike the standard put option, a Reset put option has a stochastic strike price. On issue date, the reset put option has a strike price equal to the stock price. However, if the stock price exceeds the original strike price on a pre-specified future reset date, then the strike price is reset to the current stock price.

[Gray & Whaley, 1999]

Introduction (cont.)

This exotic option provides traders an opportunity for larger payoffs compared to vanilla put options. However, this comes with a risk of more a expensive premium.

For example, if a trader is expecting the price of a stock to increase in the immediate future but decrease in the long run, then he/she would buy a reset put option today. During the reset date, if the stock price increases, he/she would have secured a higher strike price compared to a vanilla (at-the-money) put option bought today, and consequently, a higher payoff.

Reset put options can be traded as separate securities and can be found in a number of markets worldwide, such as in the Taiwan Stock Exchange, New York Stock Exchange, and the Chicago Board Exchange.

Introduction (cont.)

In [Gray & Whaley, 1999], the strike price resets only if the stock price at the reset date is higher than the original strike price. However, different types of reset (Cliquet) option exists in the markets.

Another Cliquet option considers a “series of at-the-money options, with periodic settlement, resetting the strike value at the then current price level, at which time, the option locks in the difference between the old and new strike and pays that out as the profit. The profit can be accumulated until final maturity, or paid out at each reset date.” [Shparber & Resheff, 2004]

The difference between these two types of reset (Cliquet) option is that the strike price of the latter resets regardless of whether the stock price is higher or lower than the original strike price. Because of this, we can view the exotic option as a series of forward-start options.

Reset put options can also be embedded in structured over-the-counter products. One example would be the Geared Equity Investment (GEI) from the Macquarie Bank in Australia, which is a collateralized loan plus a reset put.

Preliminaries

Preliminaries

1. The stock prices S_T follows the GBM.

$$S_T = S_0 e^{(r-q-\frac{1}{2}\sigma^2)T+\sigma W_T}.$$

If $z = (r-q-\frac{1}{2}\sigma^2)T + \sigma W_T$, then $z \sim N\left((r-q-\frac{1}{2}\sigma^2)T, \sigma^2 T\right)$.

2. For $0 \leq t_1 < t_2 \leq t_3 < t_4$,

$$W_{t_4} - W_{t_3} \perp\!\!\!\perp W_{t_2} - W_{t_1}.$$

Using a result from MATH 236.4, if f, g are Borel functions, then

$$f\left(W_{t_4} - W_{t_3}\right) \perp\!\!\!\perp g\left(W_{t_2} - W_{t_1}\right).$$

3. Covariance of Wiener process is

$$\text{Cov}(W_t, W_s) = \min(t, s).$$

Preliminaries (cont.)

4. From the lecture,

$$z - \frac{1}{2}\left[\frac{z-(r-q-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right]^2 = (r-q)T - \frac{1}{2}\left[\frac{z-(r-q+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right]^2.$$

5. It can be shown (numerically) that

$$\begin{aligned} & z_2 - \frac{1}{2}\begin{bmatrix} z_1-(r-q-\frac{1}{2}\sigma^2)T_1 \\ z_2-(r-q-\frac{1}{2}\sigma^2)T_2 \end{bmatrix}^T \begin{bmatrix} \sigma^2 T_1 & \sigma^2 T_1 \\ \sigma^2 T_1 & \sigma^2 T_2 \end{bmatrix}^{-1} \begin{bmatrix} z_1-(r-q-\frac{1}{2}\sigma^2)T_1 \\ z_2-(r-q-\frac{1}{2}\sigma^2)T_2 \end{bmatrix} \\ &= (r-q)T_2 - \frac{1}{2}\begin{bmatrix} z_1-(r-q+\frac{1}{2}\sigma^2)T_1 \\ z_2-(r-q+\frac{1}{2}\sigma^2)T_2 \end{bmatrix}^T \begin{bmatrix} \sigma^2 T_1 & \sigma^2 T_1 \\ \sigma^2 T_1 & \sigma^2 T_2 \end{bmatrix}^{-1} \begin{bmatrix} z_1-(r-q+\frac{1}{2}\sigma^2)T_1 \\ z_2-(r-q+\frac{1}{2}\sigma^2)T_2 \end{bmatrix}. \end{aligned}$$

Preliminaries (cont.)

6. The process $\{z_t\} = \left\{(r - q - \frac{1}{2}\sigma^2)t + \sigma W_t\right\}$ has independent Gaussian increments. Therefore, from MATH 236.1, $\{z_t\}$ is a Gaussian process. Thus, if $z_1 := z_{T_1}$ and $z_2 := z_{T_2}$ where $T_1 \leq T_2$, then $[z_1, z_2]^T$ has a bivariate Normal distribution. Their joint probability density function is

$$f(z_1, z_2) = \frac{1}{2\pi\sigma^2\sqrt{T_1(T_2 - T_1)}} \exp\left\{ -\frac{1}{2} \begin{bmatrix} z_1 - (r - q - \frac{1}{2}\sigma^2)T_1 \\ z_2 - (r - q - \frac{1}{2}\sigma^2)T_2 \end{bmatrix}^T \begin{bmatrix} \sigma^2 T_1 & \sigma^2 T_1 \\ \sigma^2 T_1 & \sigma^2 T_2 \end{bmatrix}^{-1} \begin{bmatrix} z_1 - (r - q - \frac{1}{2}\sigma^2)T_1 \\ z_2 - (r - q - \frac{1}{2}\sigma^2)T_2 \end{bmatrix} \right\}.$$

Preliminaries (cont.)

7. From the lectures,

$$\int_0^{\infty} e^z g(z)dz = \int_0^{\infty} \frac{1}{\sqrt{2\pi}\sigma\sqrt{T}} \exp\left\{z - \frac{1}{2}\left[\frac{z-(r-q-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right]\right\} dz = e^{(r-q)T}N(d_1),$$

where $d_1 = \dfrac{(r-q+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$. Furthermore, $\displaystyle\int_{-\infty}^{0} e^z g(z)dz = e^{(r-q)T}N(-d_1)$.

8. From the lectures,

$$\int_0^{\infty} g(z)dz = \int_0^{\infty} \frac{1}{\sqrt{2\pi}\sigma\sqrt{T}} \exp\left\{-\frac{1}{2}\left[\frac{z-(r-q-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right]\right\} dz = N(d_2),$$

where $d_2 = \dfrac{(r-q-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$. Furthermore, $\displaystyle\int_{-\infty}^{0} g(z)dz = N(-d_2)$.

Assumptions

Assumptions

The following assumptions were used in pricing the option.

1. The underlying asset of the option is a stock.
2. The stock price S_t follows GBM.
3. There is a fixed dividend yield q during the life of the option.
4. The risk-free rate of interest r is constant and the same for all maturities.
5. The time today is $T_0 = 0$.
6. The option only has one reset date at T_1 and expires at T_2.
7. The option is a European option.
8. The initial strike price is S_{T_0}, the closing stock price on the date of issuance. The payoff at maturity is

$$\left(S_{T_2} - \max(S_{T_1}, S_{T_0})\right)^+ .$$

Pricing Formula

Payoff

Let S_0 be the price of the underlying stock today T_0, S_1 be the price of the underlying stock at the reset date T_1, and S_2 be the price of the underlying stock at maturity T_2.

$$\Phi(S_2) = (S_2 - \max(S_1, S_0))^+ = \begin{cases} S_1 - S_2, & \text{if } S_1 > S_0, S_2 \leq S_1 \\ S_0 - S_2, & \text{if } S_1 \leq S_0, S_2 \leq S_0 \\ 0, & \text{Otherwise} \end{cases}.$$

Pricing Formula

Using the risk neutral valuation, the price of the option today T_0 is

$$\begin{aligned} p_0 &= e^{-rT_2} E\left[\Phi(S_2) \mid S_0 = s\right] \\ &= \underbrace{e^{-rT_2} E[S_1 - S_2 \mid S_1 > S_0, S_2 \leq S_1] P(S_1 > S_0, S_2 \leq S_1)}_{A} \\ &\qquad + \underbrace{e^{-rT_2} E[S_0 - S_2 \mid S_1 \leq S_0, S_2 \leq S_0] P(S_1 \leq S_0, S_2 \leq S_0)}_{B} \end{aligned}$$

Pricing Formula A

$$A = e^{-rT_2} E[S_1 - S_2 \mid S_1 > S_0, S_2 \leq S_1] P(S_1 > S_0, S_2 \leq S_1)$$

Let $S_1 = S_0 e^{z_1}$, where $z_1 = (r - q - \frac{1}{2}\sigma^2)T_1 + \sigma W_{T_1}$, and $S_2 = S_1 e^{z_2}$, where $z_2 = (r - q - \frac{1}{2}\sigma^2)(T_2 - T_1) + \sigma(W_{T_2} - W_{T_1})$. Since $W_{T_1} \perp\!\!\!\perp W_{T_2} - W_{T_1}$, then $z_1 \perp\!\!\!\perp z_2$.

Moreover, from the preliminaries, $[z_1, z_2]^T$ has a bivariate normal distribution. Since $z_1 \perp\!\!\!\perp z_2$, their joint pdf is the product of their pdf.

Furthermore, the region of integration for A is

$$S_1 > S_0 \implies S_0 e^{z_1} > S_0 \implies z_1 > 0,$$

$$S_2 \leq S_1 \implies S_1 e^{z_2} \leq S_1 \implies z_2 \leq 0.$$

Pricing Formula A (cont.)

Let $g(z_1), g(z_2)$ be the pdfs of z_1, z_2 respectively.

$$\begin{aligned}
A &= e^{-rT_2}\int_{-\infty}^{0}\int_{0}^{\infty}[S_1 - S_2]\cdot g(z_1)g(z_2)dz_1dz_2 \\
&= e^{-rT_2}\int_{-\infty}^{0}\int_{0}^{\infty} S_1\cdot g(z_1)g(z_2)dz_1dz_2 - e^{-rT_2}\int_{-\infty}^{0}\int_{0}^{\infty} S_2\cdot g(z_1)g(z_2)dz_1dz_2 \\
&= e^{-rT_2}\int_{-\infty}^{0}\int_{0}^{\infty} S_0e^{z_1}\cdot g(z_1)g(z_2)dz_1dz_2 - e^{-rT_2}\int_{-\infty}^{0}\int_{0}^{\infty} S_1e^{z_2}\cdot g(z_1)g(z_2)dz_1dz_2 \\
&= S_0e^{-rT_2}\int_{0}^{\infty} e^{z_1}\cdot g(z_1)dz_1\int_{-\infty}^{0} g(z_2)dz_2 - e^{-rT_2}\int_{0}^{\infty} S_1\cdot g(z_1)dz_1\int_{-\infty}^{0} e^{z_2}\cdot g(z_2)dz_2 \\
&= S_0e^{-rT_2}\int_{0}^{\infty} e^{z_1}\cdot g(z_1)dz_1\int_{-\infty}^{0} g(z_2)dz_2 - e^{-rT_2}\int_{0}^{\infty} S_0e^{z_1}\cdot g(z_1)dz_1\int_{-\infty}^{0} e^{z_2}\cdot g(z_2)dz_2 \\
&= S_0e^{-rT_2}\int_{0}^{\infty} e^{z_1}\cdot g(z_1)dz_1\int_{-\infty}^{0} g(z_2)dz_2 - S_0e^{-rT_2}\int_{0}^{\infty} e^{z_1}\cdot g(z_1)dz_1\int_{-\infty}^{0} e^{z_2}\cdot g(z_2)dz_2
\end{aligned}$$

Pricing Formula A (cont.)

$$
\begin{aligned}
&= S_0 e^{-rT_2} \int_0^{\infty} e^{z_1} \cdot g(z_1) dz_1 \int_{-\infty}^{0} g(z_2) dz_2 - S_0 e^{-rT_2} \int_0^{\infty} e^{z_1} \cdot g(z_1) dz_1 \int_{-\infty}^{0} e^{z_2} \cdot g(z_2) dz_2 \\
&= S_0 e^{-rT_2} \left[e^{(r-q)T_1} N(a_1) \right] \left[N(-c_2) \right] - S_0 e^{-rT_2} \left[e^{(r-q)T_1} N(a_1) \right] \left[e^{(r-q)(T_2-T_1)} N(-c_1) \right] \\
A &= S_0 e^{-qT_1 - r(T_2-T_1)} N(a_1) N(-c_2) - S_0 e^{-qT_2} N(a_1) N(-c_1),
\end{aligned}
$$

where $a_1 = \dfrac{(r-q+\frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T_1}}$, $a_2 = a_1 - \sigma\sqrt{T_1}$, $c_1 = \dfrac{(r-q+\frac{1}{2}\sigma^2)(T_2-T_1)}{\sigma\sqrt{T_2-T_1}}$, $c_2 = c_1 - \sigma\sqrt{T_2-T_1}$.

Pricing Formula B

$$B = e^{-rT_2} E[S_0 - S_2 \mid S_1 \leq S_0, S_2 \leq S_0] P(S_1 \leq S_0, S_2 \leq S_0)$$

Let $S_1 = S_0 e^{z_1}$, where $z_1 = (r - q - \frac{1}{2}\sigma^2)T_1 + \sigma W_{T_1}$, and $S_2 = S_0 e^{z_2}$, where $z_2 = (r - q - \frac{1}{2}\sigma^2)T_2 + \sigma W_{T_2}$. Note that z_1 and z_2 are not independent.

$$\begin{aligned}
\text{Cov}(z_1, z_2) &= \text{Cov}\left((r - q - \frac{1}{2}\sigma^2)T_1 + \sigma W_{T_1}, (r - q - \frac{1}{2}\sigma^2)T_2 + \sigma W_{T_2}\right) \\
&= \text{Cov}(\sigma W_{T_1}, \sigma W_{T_2}) \\
&= \sigma^2 \text{Cov}(W_{T_1}, W_{T_2}) \\
&= \sigma^2 T_1 \\
\rho &= \frac{\text{Cov}(z_1, z_2)}{\sqrt{\text{Var}(z_1)}\sqrt{\text{Var}(z_2)}} \\
&= \frac{\sigma^2 T_1}{\sigma\sqrt{T_1}\sigma\sqrt{T_2}} = \sqrt{\frac{T_1}{T_2}}
\end{aligned}$$

Pricing Formula B (cont.)

Moreover, from the preliminaries, $[z_1, z_2]^T$ has a bivariate normal distribution. The joint pdf is

$$f(z_1, z_2) = \frac{1}{2\pi\sigma^2\sqrt{T_1(T_2 - T_1)}} \exp\left\{ -\frac{1}{2} \begin{bmatrix} z_1 - (r - q - \frac{1}{2}\sigma^2)T_1 \\ z_2 - (r - q - \frac{1}{2}\sigma^2)T_2 \end{bmatrix}^T \begin{bmatrix} \sigma^2 T_1 & \sigma^2 T_1 \\ \sigma^2 T_1 & \sigma^2 T_2 \end{bmatrix}^{-1} \begin{bmatrix} z_1 - (r - q - \frac{1}{2}\sigma^2)T_1 \\ z_2 - (r - q - \frac{1}{2}\sigma^2)T_2 \end{bmatrix} \right\}$$

Furthermore, the region of integration for B is

$$S_1 \leq S_0 \implies S_0 e^{z_1} \leq S_0 \implies z_1 \leq 0,$$

$$S_2 \leq S_0 \implies S_0 e^{z_2} \leq S_0 \implies z_2 \leq 0.$$

Pricing Formula B (cont.)

Let $f(z_1, z_2)$ be the joint pdf of z_1 and z_2.

$$\begin{aligned}
B &= e^{-rT_2} \int_{-\infty}^{0} \int_{-\infty}^{0} [S_0 - S_2] \cdot f(z_1, z_2) dz_1 dz_2 \\
&= e^{-rT_2} \int_{-\infty}^{0} \int_{-\infty}^{0} S_0 \cdot f(z_1, z_2) dz_1 dz_2 - e^{-rT_2} \int_{-\infty}^{0} \int_{-\infty}^{0} S_2 \cdot f(z_1, z_2) dz_1 dz_2 \\
&= S_0 e^{-rT_2} \int_{-\infty}^{0} \int_{-\infty}^{0} f(z_1, z_2) dz_1 dz_2 - e^{-rT_2} \int_{-\infty}^{0} \int_{-\infty}^{0} S_0 e^{z_2} \cdot f(z_1, z_2) dz_1 dz_2 \\
&= S_0 e^{-rT_2} \underbrace{\int_{-\infty}^{0} \int_{-\infty}^{0} f(z_1, z_2) dz_1 dz_2}_{B_1} - S_0 e^{-rT_2} \underbrace{\int_{-\infty}^{0} \int_{-\infty}^{0} e^{z_2} \cdot f(z_1, z_2) dz_1 dz_2}_{B_2}
\end{aligned}$$

Pricing Formula B_1

$$\begin{aligned}
B_1 &= \int_{-\infty}^{0}\int_{-\infty}^{0} f(z_1,z_2)dz_1dz_2\\
&= \int_{-\infty}^{0}\int_{-\infty}^{0} \frac{1}{2\pi\sigma^2\sqrt{T_1(T_2-T_1)}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{z_1-(r-q-\frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T_1}}\right)^2\right.\right.\\
&\qquad \left.\left.-2\rho\left(\frac{z_1-(r-q-\frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T_1}}\right)\left(\frac{z_2-(r-q-\frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}}\right)+\left(\frac{z_2-(r-q-\frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}}\right)^2\right]\right\}dz_1dz_2
\end{aligned}$$

Let $u_1 = \dfrac{z_1-(r-q-\frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T_1}}$. Thus, $du_1 = \dfrac{dz_1}{\sigma\sqrt{T_1}}$. Also, if $z_1 = 0$, $u_1 = -\dfrac{(r-q-\frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T_1}}$ and $u_1 \to -\infty$ as $z_1 \to -\infty$.

$$\begin{aligned}
&= \int_{-\infty}^{0}\int_{-\infty}^{-a_2} \frac{1}{2\pi\sigma\sqrt{T_2-T_1}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[u_1^2-2\rho u_1\left(\frac{z_2-(r-q-\frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}}\right)\right.\right.\\
&\qquad \left.\left.+\left(\frac{z_2-(r-q-\frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}}\right)^2\right]\right\}du_1dz_2,
\end{aligned}$$

where $a_2 = \dfrac{(r-q-\frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T_1}}$.

Pricing Formula B_1 (cont.)

$$= \int_{-\infty}^{0} \int_{-\infty}^{-a_2} \frac{1}{2\pi\sigma\sqrt{T_2 - T_1}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[u_1^2 - 2\rho u_1 \left(\frac{z_2 - (r - q - \frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}} \right) + \left(\frac{z_2 - (r - q - \frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}} \right)^2 \right] \right\} du_1 dz_2$$

Let $u_2 = \frac{z_2 - (r - q - \frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}}$. Thus, $du_2 = \frac{dz_2}{\sigma\sqrt{T_2}}$. Also, if $z_2 = 0$, $u_2 = -\frac{(r - q - \frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}}$ and $u_2 \to -\infty$ as $z_2 \to -\infty$.

$$= \int_{-\infty}^{-b_2} \int_{-\infty}^{-a_2} \frac{1}{2\pi\sqrt{1 - \frac{T_1}{T_2}}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[u_1^2 - 2\rho u_1 u_2 + u_2^2 \right] \right\} du_1 du_2$$

$$= \int_{-\infty}^{-b_2} \int_{-\infty}^{-a_2} \frac{1}{2\pi\sqrt{1 - \rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[u_1^2 - 2\rho u_1 u_2 + u_2^2 \right] \right\} du_1 du_2 = N_2\left(-b_2, -a_2, \sqrt{\frac{T_1}{T_2}} \right),$$

where $b_2 = \frac{(r - q - \frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}}$ and $N_2(a, b, \rho)$ is a cumulative bivariate normal distribution function with upper integral limits a and b and correlation coefficient ρ.

Pricing Formula B_2

$$B_2 = \int_{-\infty}^{0}\int_{-\infty}^{0} e^{z_2} \cdot f(z_1, z_2) dz_1 dz_2$$

$$= \int_{-\infty}^{0}\int_{-\infty}^{0} \frac{1}{2\pi\sigma^2\sqrt{T_1(T_2 - T_1)}} \exp\Bigg\{ z_2 - \frac{1}{2(1-\rho^2)}\Bigg[\left(\frac{z_1 - (r - q - \frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T_1}}\right)^2$$

$$- 2\rho \left(\frac{z_1 - (r - q - \frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T_1}}\right)\left(\frac{z_2 - (r - q - \frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}}\right) + \left(\frac{z_2 - (r - q - \frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}}\right)^2 \Bigg]\Bigg\} dz_1 dz_2$$

From preliminaries,

$$= \int_{-\infty}^{0}\int_{-\infty}^{0} \frac{1}{2\pi\sigma^2\sqrt{T_1(T_2 - T_1)}} \exp\Bigg\{ (r-q)T_2 - \frac{1}{2(1-\rho^2)}\Bigg[\left(\frac{z_1 - (r - q + \frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T_1}}\right)^2$$

$$- 2\rho \left(\frac{z_1 - (r - q + \frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T_1}}\right)\left(\frac{z_2 - (r - q + \frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}}\right) + \left(\frac{z_2 - (r - q + \frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}}\right)^2 \Bigg]\Bigg\} dz_1 dz_2$$

$$= e^{(r-q)T_2} \int_{-\infty}^{0}\int_{-\infty}^{0} \frac{1}{2\pi\sigma^2\sqrt{T_1(T_2 - T_1)}} \exp\Bigg\{ - \frac{1}{2(1-\rho^2)}\Bigg[\left(\frac{z_1 - (r - q + \frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T_1}}\right)^2$$

$$- 2\rho \left(\frac{z_1 - (r - q + \frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T_1}}\right)\left(\frac{z_2 - (r - q + \frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}}\right) + \left(\frac{z_2 - (r - q + \frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}}\right)^2 \Bigg]\Bigg\} dz_1 dz_2$$

Pricing Formula B_2 (cont.)

$$= e^{(r-q)T_2} \int_{-\infty}^{0} \int_{-\infty}^{0} \frac{1}{2\pi\sigma^2\sqrt{T_1(T_2-T_1)}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[\left(\frac{z_1 - (r-q+\frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T_1}} \right)^2 \right.\right.$$
$$\left.\left. - 2\rho \left(\frac{z_1 - (r-q+\frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T_1}} \right) \left(\frac{z_2 - (r-q+\frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}} \right) + \left(\frac{z_2 - (r-q+\frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}} \right)^2 \right] \right\} dz_1\, dz_2$$

Let $u_1 = \frac{z_1 - (r-q+\frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T_1}}$. Thus, $du_1 = \frac{dz_1}{\sigma\sqrt{T_1}}$. Also, if $z_1 = 0$, $u_1 = -\frac{(r-q+\frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T_1}}$ and $u_1 \to -\infty$ as $z_1 \to -\infty$.

$$= e^{(r-q)T_2} \int_{-\infty}^{0} \int_{-\infty}^{-a_1} \frac{1}{2\pi\sigma\sqrt{T_2-T_1}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[u_1^2 - 2\rho u_1 \left(\frac{z_2 - (r-q+\frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}} \right) \right.\right.$$
$$\left.\left. + \left(\frac{z_2 - (r-q+\frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}} \right)^2 \right] \right\} du_1\, dz_2,$$

where $a_1 = \frac{(r-q+\frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T_1}}$.

Pricing Formula B_2 (cont.)

$$= e^{(r-q)T_2} \int_{-\infty}^{0} \int_{-\infty}^{-a_1} \frac{1}{2\pi\sigma\sqrt{T_2 - T_1}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[u_1^2 - 2\rho u_1 \left(\frac{z_2 - (r - q + \frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}} \right) + \left(\frac{z_2 - (r - q + \frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}} \right)^2 \right] \right\} du_1\, dz_2$$

Let $u_2 = \dfrac{z_2 - (r - q + \frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}}$. Thus, $du_2 = \dfrac{dz_2}{\sigma\sqrt{T_2}}$. Also, if $z_2 = 0$, $u_2 = -\dfrac{(r - q + \frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}}$ and $u_2 \to -\infty$ as $z_2 \to -\infty$.

$$= e^{(r-q)T_2} \int_{-\infty}^{-b_1} \int_{-\infty}^{-a_1} \frac{1}{2\pi\sqrt{1 - \frac{T_1}{T_2}}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[u_1^2 - 2\rho u_1 u_2 + u_2^2 \right] \right\} du_1\, dz_2$$

$$= e^{(r-q)T_2} \int_{-\infty}^{-b_1} \int_{-\infty}^{-a_1} \frac{1}{2\pi\sqrt{1 - \rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[u_1^2 - 2\rho u_1 u_2 + u_2^2 \right] \right\} du_1\, dz_2 = e^{(r-q)T_2} N_2\left(-b_1, -a_1, \sqrt{\frac{T_1}{T_2}} \right),$$

where $b_1 = \dfrac{(r - q + \frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}}$ and $N_2(a, b, \rho)$ is a cumulative bivariate normal distribution function with upper integral limits a and b and correlation coefficient ρ.

Pricing Formula B (cont.)

Therefore,

$$\begin{aligned}B &= S_0 e^{-rT_2} \int_{-\infty}^{0} \int_{-\infty}^{0} f(z_1, z_2) dz_1 dz_2 - S_0 e^{-rT_2} \int_{-\infty}^{0} \int_{-\infty}^{0} e^{z_2} \cdot f(z_1, z_2) dz_1 dz_2 \\ &= S_0 e^{-rT_2} N_2 \left(-b_2, -a_2, \sqrt{\frac{T_1}{T_2}} \right) - S_0 e^{-rT_2} \left[e^{(r-q)T_2} N_2 \left(-b_1, -a_1, \sqrt{\frac{T_1}{T_2}} \right) \right] \\ &= S_0 e^{-rT_2} N_2 \left(-b_2, -a_2, \sqrt{\frac{T_1}{T_2}} \right) - S_0 e^{-qT_2} N_2 \left(-b_1, -a_1, \sqrt{\frac{T_1}{T_2}} \right),\end{aligned}$$

where $a_1 = \dfrac{(r - q + \frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T_1}}$, $a_2 = a_1 - \sigma\sqrt{T_1}$, $b_1 = \dfrac{(r - q + \frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}}$, $b_2 = b_1 - \sigma\sqrt{T_2}$.

Final Pricing Formula

Therefore, the price of the Reset Put option at initiation is

$$p_0 = S_0 e^{-qT_1 - r(T_2 - T_1)} N(a_1) N(-c_2) - S_0 e^{-qT_2} N(a_1) N(-c_1)$$
$$+ S_0 e^{-rT_2} N_2\left(-b_2, -a_2, \sqrt{\frac{T_1}{T_2}}\right) - S_0 e^{-qT_2} N_2\left(-b_1, -a_1, \sqrt{\frac{T_1}{T_2}}\right),$$

where $a_1 = \dfrac{(r - q + \frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T_1}}, a_2 = a_1 - \sigma\sqrt{T_1},$

$$b_1 = \frac{(r - q + \frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}}, b_2 = b_1 - \sigma\sqrt{T_2},$$

$$c_1 = \frac{(r - q + \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}}, c_2 = c_1 - \sigma\sqrt{T_2 - T_1}.$$

Extensions

Extensions

Monte Carlo: In [Shparber & Resheff, 2004], a Monte Carlo simulation was used to price the reset put option. However, their simulation was based on the binomial tree model rather than the discretized GBM approach.

Heston's Model: In [Detlefsen & Härdle, 2006], Heston's model was used alongside Monte Carlo simulation to price the reset put option.

Binomial Tree: A naive approach to the binomial tree would be to track the path taken by each trajectory. This would lead to an exponential complexity. In [Shparber & Resheff, 2004], an optimized binomial tree solution was proposed. To price the exotic option,

$$e^{-rT_2}\sum_{j=0}^{n_1}\sum_{i=j}^{n_2-n_1+j}\frac{n_1!(n_2-n_1)!}{j!(n_1-j)!(i-j)!(n_2-n_1-i+j)!}p^i(1-p)^{n_2-i}\max(S_0-S_0u^id^{n_2-i},S_0u^jd^{n_1-j}-S_0u^id^{n_2-i},0).$$

A Python implementation of Monte Carlo and Binomial Tree can be found here.

Example

Example 1

European Reset Put

Calculate the price of a European Reset Put option on a stock when the initial price is $100, the risk-free interest rate is 10% per annum, the dividends yield is 5% per annum, and the volatility is 30% per annum, and the time to maturity is 1 year where the reset date is in 6 months.

First, by the closed form that we have previously derived, the price of the European Reset put option is given as:

$$p_0 = S_0 e^{-qT_1 - r(T_2 - T_1)} N(a_1) N(-c_2) - S_0 e^{-qT_2} N(a_1) N(-c_1)$$
$$+ S_0 e^{-rT_2} N_2\left(-b_2, -a_2, \sqrt{\frac{T_1}{T_2}}\right) - S_0 e^{-qT_2} N_2\left(-b_1, -a_1, \sqrt{\frac{T_1}{T_2}}\right).$$

Using Python, (code in slide 50), we get $p_0 = 11.5096$, where
$a_1 = 0.2239$, $a_2 = 0.0118$, $b_1 = 0.3167$, $b_2 = 0.0167$, $c_1 = 0.2239$, $c_2 = 0.0118$.

Example 1 (cont.)

European Reset Put

Calculate the price of a European Reset Put option on a stock when the initial price is \$100, the risk-free interest rate is 10% per annum, the dividends yield is 5% per annum, and the volatility is 30% per annum, and the time to maturity is 1 year where the reset date is in 6 months.

Next, using Monte Carlo simulation on the discretized GBM, we simulated different trajectories and obtained the discounted mean. The discretized GBM is

$$S_{t+\Delta t} = S_t \left[1 + (r - q)\Delta t + \sigma \sqrt{\Delta t} Z \right],$$

where $Z \sim N(0,1)$. Using Python, (code in slide 49), we get $p_0 = 11.5114$, where we used 1,000,000 simulations.

Example 1 (cont.)

European Reset Put

Calculate the price of a European Reset Put option on a stock when the initial price is \$100, the risk-free interest rate is 10% per annum, the dividends yield is 5% per annum, and the volatility is 30% per annum, and the time to maturity is 1 year where the reset date is in 6 months.

Finally, using an optimized binomial tree approach by [Shparber & Resheff, 2004], the price of the exotic option is

$$e^{-rT_2}\sum_{j=0}^{n_1}\sum_{i=j}^{n_2-n_1+j}\frac{n_1!(n_2-n_1)!}{j!(n_1-j)!(i-j)!(n_2-n_1-i+j)!}p^i(1-p)^{n_2-i}\max(S_0-S_0u^id^{n_2-i},S_0u^jd^{n_1-j}-S_0u^id^{n_2-i},0),$$

where $p_u=\dfrac{e^{(r-q)\Delta t}-d}{u-d}=0.5003, p_d=1-p_u=0.4997, u=e^{\sigma\sqrt{\Delta t}}=1.0095, d=\dfrac{1}{u}=0.9906$. Using Python, (code in slide 47), we get $p_0=11.5039$, where we used 1,000 steps.

Example 2

European Reset Put

What is the price of a European Reset put option on a non-dividend-paying stock when the stock price is $60, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is in 6 months with pre-specified reset date in 2 months.

By the closed form:

$$p_0 = S_0 e^{-qT_1 - r(T_2 - T_1)} N(a_1) N(-c_2) - S_0 e^{-qT_2} N(a_1) N(-c_1) + S_0 e^{-rT_2} N_2\left(-b_2, -a_2, \sqrt{\frac{T_1}{T_2}}\right) - S_0 e^{-qT_2} N_2\left(-b_1, -a_1, \sqrt{\frac{T_1}{T_2}}\right).$$

Using Python, (code in slide 50), we get $p_0 = 6.3845$, where
$a_1 = 0.1298, a_2 = -0.0131, b_1 = 0.2248, b_2 = -0.0227, c_1 = 0.1835, c_2 = -0.0186.$

Example 2 (cont.)

European Reset Put

What is the price of a European Reset put option on a non-dividend-paying stock when the stock price is $60, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is in 6 months with pre-specified reset date in 2 months.

Using Monte Carlo simulation on the discretized GBM, we simulated different trajectories and obtained the discounted mean. The discretized GBM is

$$S_{t+\Delta t} = S_t \left[1 + (r - q)\Delta t + \sigma\sqrt{\Delta t}Z\right],$$

where $Z \sim N(0,1)$. Using Python, (code in slide 49), we get $p_0 = 6.4841$, using 1,000,000 simulations.

Example 2 (cont.)

European Reset Put

What is the price of a European Reset put option on a non-dividend-paying stock when the stock price is \$60, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is in 6 months with pre-specified reset date in 2 months.

Lastly, by the binomial tree approach, we get

$$e^{-rT_2}\sum_{j=0}^{n_1}\sum_{i=j}^{n_2-n_1+j}\frac{n_1!(n_2-n_1)!}{j!(n_1-j)!(i-j)!(n_2-n_1-i+j)!}p^i(1-p)^{n_2-i}\max(S_0-S_0u^id^{n_2-i},S_0u^jd^{n_1-j}-S_0u^id^{n_2-i},0),$$

where $p_u=\dfrac{e^{(r-q)\Delta t}-d}{u-d}=0.4996, p_d=1-p_u=0.5004, u=e^{\sigma\sqrt{\Delta t}}=1.0079, d=\dfrac{1}{u}=0.9922.$
Using Python, (code in slide 49), we get $p_0=6.4750$, using 1,000 steps.

Summary and Conclusion

Summary

In summary, the closed form of the reset put option proposed by [Gray & Whaley, 1999] is

$$p_0 = S_0 e^{-qT_1 - r(T_2 - T_1)} N(a_1) N(-c_2) - S_0 e^{-qT_2} N(a_1) N(-c_1)$$
$$+ S_0 e^{-rT_2} N_2\left(-b_2, -a_2, \sqrt{\frac{T_1}{T_2}}\right) - S_0 e^{-qT_2} N_2\left(-b_1, -a_1, \sqrt{\frac{T_1}{T_2}}\right),$$

where $a_1 = \dfrac{(r - q + \frac{1}{2}\sigma^2) T_1}{\sigma\sqrt{T_1}}, a_2 = a_1 - \sigma\sqrt{T_1},$

$$b_1 = \frac{(r - q + \frac{1}{2}\sigma^2) T_2}{\sigma\sqrt{T_2}}, b_2 = b_1 - \sigma\sqrt{T_2},$$

$$c_1 = \frac{(r - q + \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}}, c_2 = c_1 - \sigma\sqrt{T_2 - T_1}.$$

Summary (cont.)

In practice, most reset options, including all of the reset options listed in TSE, CBOE, and NYSE, are options with multiple strike resets and reset dates. Hence, similar (numerical) valuation techniques demonstrated, such as the binomial tree and Monte Carlo simulation, can also be employed in valuing these exotic options as it would be more difficult to find a closed form solution. The use of numerical techniques has a trade-off between precision and efficiency. In general, more steps/simulations are required to achieve better results.

[Liao & Wang, 2002]

Overall, the reset put option is a type of exotic option that provides traders who are bearish on a stock an opportunity for greater payoffs if the stock price unexpectedly rises during the reset date. Because of this, reset options are generally more expensive than vanilla put options. However, if at the reset date, the strike price did not reset, then they would have incurred an additional loss. Greater risk management are needed for traders who wish to trade this exotic option since they are more sensitive to changes compared to vanilla options.

[Gray & Whaley, 1999]

References

References

- Detlefsen, K., & Härdle, W. K. K. (2006). Calibration Risk for Exotic Options (Vol. 14). https://www.researchgate.net/publication/5101363_Calibration_Risk_for_Exotic_Options
- Gray, S., & Whaley, R. (1999). Reset Put Options: Valuation, Risk Characteristics, and an Application. Australian Journal of Management, 24(1). https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.1015.8221&rep=rep1&type=pdf
- Liao, S.-L., & Wang, C.-W. (2002). The Valuation of Reset Options with Multiple Strike Resets and Reset Dates. http://www3.nccu.edu.tw/ liaosl/Publication/17.pdf
- Shparber, M., & Resheff, S. (2004). Valuation of Cliquet Options. https://www.researchgate.net/publication/240390357_Valuation_of_Cliquet_Options/citation/download

Appendix

Python Implementation of Preliminaries 5

```
def eq1(z, r, q, sigma, T):
    mu = np.array([
        ( r - q - 0.5 * sigma**2 ) * T[0],
        ( r - q - 0.5 * sigma**2 ) * T[1]
    ])
    Sigma = np.array([
        [sigma**2 * T[0], sigma**2 * T[0]],
        [sigma**2 * T[0], sigma**2 * T[1]]
    ])

    return z[1] - 0.5 * np.dot( (z - mu).T, np.dot( np.linalg.inv(Sigma), (z - mu) ) )
```

Python Implementation of Preliminaries 5

```
def eq2(z, r, q, sigma, T):
    mu = np.array([
        ( r - q + 0.5 * sigma**2 ) * T[0],
        ( r - q + 0.5 * sigma**2 ) * T[1]
    ])
    Sigma = np.array([
        [sigma**2 * T[0], sigma**2 * T[0]],
        [sigma**2 * T[0], sigma**2 * T[1]]
    ])

    return ( (r - q) * T[1] -
        0.5 * np.dot( (z - mu).T, np.dot( np.linalg.inv(Sigma), (z - mu) ) ) )
```

Python Implementation of Combination Function

```
memo = [[-1 for _ in range(1000)] for _ in range(1000)]

def choose(n,r):
    if r > n or r < 0:
        return 0
    elif n == 1:
        return 1
    elif memo[n][r] == -1:
        memo[n][r] = choose(n-1, r-1) + choose(n-1, r)
    return memo[n][r]
```

Python Implementation of Binomial Tree Approach

```
def BT_option_price(S0, T2, n1, n2, r, q, sigma):
    dt = T2 / n2
    u = np.exp(  sigma * np.sqrt(dt));                  d = 1 / u
    pu = (np.exp( (r - q) * dt) - d) / (u - d);     pd = 1 - pu
    ans = 0
    for j in range(n1+1):
        for i in range(j, n2 - n1 + j + 1):
            ans += (
                choose(n1, j) * choose (n2 - n1, i - j) * pow(pu, i) * pow(pd, n2 - i) *
                max(
                    S0 - S0 * pow(u, i) * pow(d, n2 - i),
                    S0 * pow(u, j) * pow(d, n1 - j) - S0 * pow(u, i) * pow(d, n2 - i),
                    0
                )
            )
    return np.exp(-r * T2) * ans
```

Python Implementation of Monte Carlo GBM

```
def generate_trajectory(S0, T2, n2, r, q, sigma):
    dt = T2 / n2
    trajectory = [S0]
    for _ in range (n2):
        trajectory.append(
            trajectory[-1] * (1 + ( r - q ) * dt + sigma * np.sqrt(dt) * np.random.randn())
        )
    return trajectory
```

Python Implementation of Monte Carlo Approach

```
def MC_option_price(S0, T2, n1, n2, r, q, sigma):
    res = []
    for _ in range(10000):
        tmp = generate_trajectory(S0, T2, n2, r, q, sigma)
        res.append( max(max(S0, tmp[n1]) - tmp[-1] ,0) )
        plt.plot(np.array(range(n2 + 1)) / n2, tmp)
    return np.exp(- r * T2) * np.mean(res)
```

Python Implementation of Closed Form Approach

```
def closed_form_option_price(S0, T1, T2, r, q, sigma):
    a1 = ( (r - q + 0.5 * sigma**2) * T1 ) / (sigma * np.sqrt(T1))
    b1 = ( (r - q + 0.5 * sigma**2) * T2 ) / (sigma * np.sqrt(T2))
    c1 = ( (r - q + 0.5 * sigma**2) * (T2 - T1) ) / (sigma * np.sqrt(T2 - T1))
    a2 = a1 - sigma * np.sqrt(T1);      b2 = b1 - sigma * np.sqrt(T2)
    c2 = c1 - sigma * np.sqrt(T2 - T1)
    rho = np.sqrt(T1 / T2)
    Sigma = np.array([[1, rho],
                      [rho, 1]])
    return (
        S0 * np.exp(-q * T1 - r * (T2 - T1)) * norm.cdf(a1) * norm.cdf(-c2)
        - S0 * np.exp(-q * T2) * norm.cdf(a1) * norm.cdf(-c1)
        + S0 * np.exp(-r * T2) * mvn.cdf(np.array([-b2, -a2]),
                                    mean = np.zeros(2), cov = Sigma)
        - S0 * np.exp(-q * T2) * mvn.cdf(np.array([-b1, -a1]),
                                    mean = np.zeros(2), cov = Sigma)
    )
```